THERMOELASTICITY MODELS TAKING ACCOUNT OF A FINITE HEAT PROPAGATION RATE

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An analysis is made of the stationary wave process in a one-dimensional thermoelastic medium on the basis of two heat-conduction models. The nature of the change in the phase velocities, the damping coefficients, the coefficient of connectedness, and the phase-shift angle are established.

Two approaches are known in thermoelasticity theory, which result in a finite rate of heat propagation. Extensively used in one is a modified Fourier law which takes account of the heat flux relaxation [1-3], while the governing equations in the other are derived directly from the generalized inequalities of entropy production [4, 5]. Such a generalization not only results in a finite heat-propagation rate but also contains a symmetric heat-conduction tensor to linear theory accuracy [6]. We later call such models the Green — Laws models [5].

Substantially, these approaches describe different aspects of a complex process. The first approach results in a process with a quite definite relaxation, similar to the Maxwell model in the theory of viscoelasticity, and the second model is analogous to the Voigt model in the theory of viscoelasticity with elements of creep. It is clear that a more general model of thermoelasticity should contain both physical phenomena.

One-dimensional wave propagation to the accuracy of both models is studied in this paper. By assuming that the heat-conduction equations for the unconnected problem agree to the accuracy of both models in form, we obtain a single characteristic equation. In the case of the Green — Laws model, the coefficients of this equation contain two parameters m_* and n_*, which characterize the relaxation process and the rate of temperature change. In the case of using the modified Fourier law $m_* = n_*$.

The characteristic equation is solved numerically for waves with fixed frequency and length. Asymptotic formulas are also obtained to determine the phase velocities and damping coefficients. The nature of the wave is analyzed for different values of the reduced frequencies or lengths under the assumption that the phase velocities and damping coefficients are smooth functions of their arguments. A wave for which the asymptotic formulas of the phase velocity govern the velocity of the purely elastic wave in the case of the unconnected problem is hence called a modified elastic wave. Another type of wave is correspondingly called a modified thermal wave. It is shown that, depending on the properties of the medium, the velocity of a definite wave can alter the asymptotic behavior for different values of the reduced frequencies. This is the distinction from the definitions taken in [7].

1. Fundamental Equations

In the one-dimensional case in the absence of volume forces and point sources, the modified Fourier law results in the system of equations [8]

$$\begin{gathered} c_0^2 u'' - \varkappa \rho_0^{-1}\theta' = u^{\cdot\cdot}, \\ \tau_0\theta^{\cdot\cdot} + \theta^{\cdot} - k_0(\rho_0 c_e)^{-1}\theta'' + \varkappa T_0(\rho_0 c_e)^{-1}(\tau_0 u^{\cdot\cdot\prime} + u^{\cdot\prime}) = 0, \\ c_0^2 = (\lambda + 2\mu)\rho_0^{-1}, \quad \varkappa = (3\lambda + 2\mu)\alpha_T, \quad \theta = T - T_0. \end{gathered} \tag{1.1}$$

Here differentiation with respect to the Lagrange coordinate x is denoted by a prime and with respect to the time t by a point.

Moreover, the classical Clausius — Duhem inequality results in a finite heat-propagation velocity only under additional hypotheses but not in the case of a theory linearized relative to the temperature [9, 10]. Hence, the generalized inequality

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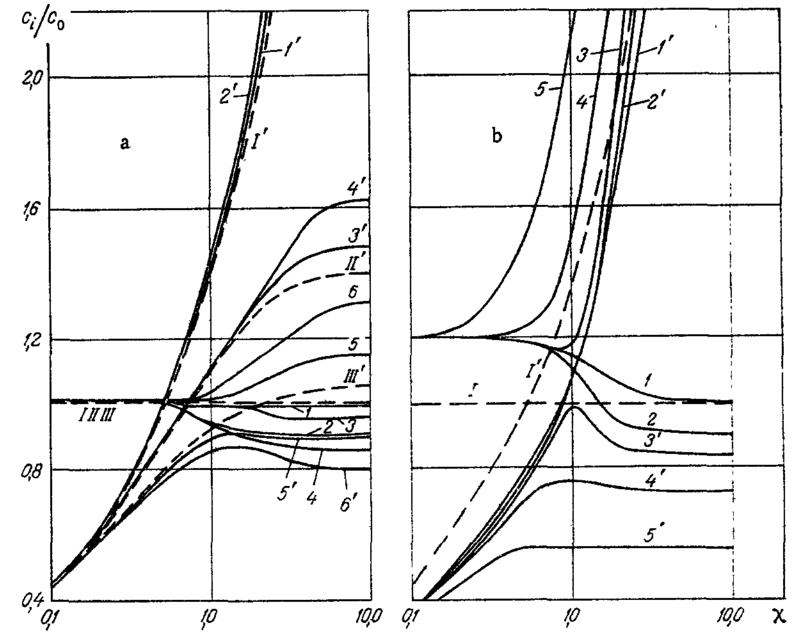

Fig. 1. Dependences of the phase velocities $c_i = c_0\chi(\mathrm{Re}\,\xi_i)^{-1}$ on the dimensionless frequency $\chi = \omega\,\omega_*^{-1}$ a) $e = 0$: I) $n_* = 0$, II) $n_* = 0.5$, III) $n_* = 0.9$; $e = 1.14 \cdot 10^{-2}$: 1) $n_* = 0$, $m_* = 5.0$; 2) $n_* = 0$, $m_* = 20.0$; 3) $n_* = 0.5$, $m_* = 5.0$; 4) $n_* = 0.5$, $m_* = 20.0$; 5) $n_* = 0.9$, $m_* = 5.0$; 6) $n_* = 0.9$, $m_* = 20.0$; b) $e = 0$: I) $n_* = 0$; $e = 0.432$, $n_* = 0$: 1) $m_* = 0$; 2) $m_* = 0.5$; 3) $m_* = 1.0$; 4) $m_* = 2.0$; 5) $m_* = 5.0$. The primed numbers are curves corresponding to the phase velocity c_2.

$$\frac{d}{dt}\int_V \rho S dV - \int_V \frac{\rho r}{\Phi} dV + \int_F \frac{q_i n_i}{\Phi} dF \geqslant 0 \tag{1.2}$$

is proposed in [5], which holds for every volume V bounded by the closed surface F of a body, where n_i is the external normal to the surface F and Φ is a scalar function. If the function Φ is given as an equation of state, then the heat-conduction law is a corollary of inequality (1.2). In the case of static processes $\Phi = T$. The Green – Laws model, obtained on the basis of (1.2), reduces to the system of equations

$$\begin{gathered} c_0^2 u'' - \varkappa\rho_0^{-1}(\theta' + \alpha\dot{\theta}') = \ddot{u}, \\ \alpha_0\ddot{\theta} + \dot{\theta} - k_0(\rho_0 c_e)^{-1}\theta'' + \varkappa T_0(\rho_0 c_e)^{-1}\dot{u}' = 0. \end{gathered} \tag{1.3}$$

In contrast to (1.1), new parameters α and α_0 are contained in (1.3), which are related by the inequality $\alpha \geq \alpha_0 \geq 0$ resulting from (1.2).

Let us make the assumption that the equations of motion and of heat conduction resulting from (1.1) and (1.3) should agree for the unconnected problem ($\varkappa = 0$). Then $\tau_0 \equiv \alpha_0$.

Let us examine plane wave propagation

$$u = u^0 \exp[i(\eta x - \omega t)], \quad \theta = \theta^0 \exp[i(\eta x - \omega t)]. \tag{1.4}$$

Taking account of (1.4), we obtain a characteristic equation [8] from the system of equations (1.1):

$$\begin{gathered} (\xi^2 - \chi^2)(\chi - in_*\chi^2 + i\xi^2) + e\xi^2\chi(1 - in_*\chi) = 0, \\ \chi = \omega\omega_*^{-1}, \quad \xi = c_0\eta\omega_*^{-1}, \quad \omega_* = c_0^2\rho_0 c_e k_0^{-1}, \\ n_* = \tau_0\omega_*, \quad e = \varkappa^2 T_0 c_0^{-2}\rho_0^{-2}c_e^{-1}. \end{gathered} \tag{1.5}$$

Analogously, we obtain from system (1.3)

$$(\xi^2 - \chi^2)(\chi - in_*\chi^2 + i\xi^2) + e\xi^2\chi(1 - im_*\chi) = 0, \tag{1.6}$$

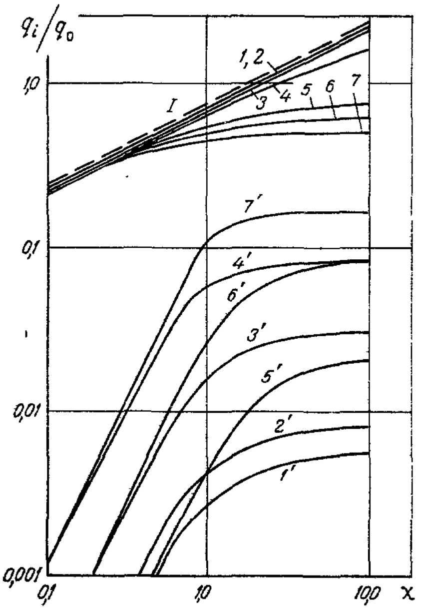

Fig. 2

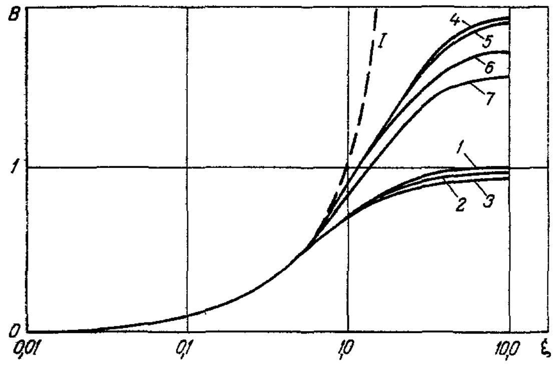

Fig. 3

Fig. 2. Dependence of the damping coefficients $q_i = q_0 \operatorname{Im} \xi_i$, $q_0 = \omega_* c_0^{-1}$ on the dimensionless frequency $\chi = \omega\, \omega_*^{-1}$: e = 0: I) $n_* = 0$; $e = 1.14 \cdot 10^{-2}$: 1) $n_* = 0$, $m_* = 0$; 2) $n_* = 0$, $m_* = 0.5$; 3) $n_* = 0$, $m_* = 5.0$; 4) $n_* = 0$, $m_* = 20.0$; 5) $n_* = 0.5$, $m^* = 0.5$; 6) $n_* = 0.5$, $m_* = 5.0$; 7) $n_* = 0.5$, $m_* = 20.0$. The primed numbers are curves corresponding to the damping coefficient q_2.

Fig. 3. Dependence of the coefficient B on the dimensionless wavelength $\xi = c_0 \eta \omega_*^{-1}$: $e = 1.14 \cdot 10^{-2}$: 1) $n_* = 0.01$, $m_* = 0.01$; 2) $n_* = 0.01$, $m_* = 5.0$; 3) $n_* = 0.01$, $m_* = 10.0$; 4) $n_* = 0.5$, $m_* = 0.5$; 5) $n_* = 0.5$, $m_* = 1.0$; 6) $n_* = 0.5$, $m_* = 5.0$; 7) $n_* = 0.5$, $m_* = 10.0$. The dashed line I corresponds to a change of the coefficient A from [8].

where $m_* = \alpha\omega_*$ and the inequality $m_* \geq n_* \geq 0$ is satisfied. Equations (1.5) and (1.6) agree when $m_* = n_*$. Then the phase velocities for both models $[c_i = c_0 \chi (\operatorname{Re} \xi_i)^{-1}]$ and the appropriate damping coefficients $[q_i = q_0 \operatorname{Im} \xi_i$, $q_0 = \omega_* c_0^{-1}$ $(i = 2, 3, 4)]$ of waves with a fixed frequency are also equal.

We obtain from (1.1) for waves with a fixed frequency [8]

$$u = u^0 \exp[-\omega_* g t + \omega_* \xi c_0^{-1} (x - f \xi^{-1} c_0 t)],$$
$$\theta = u^0 \varkappa T_0 c_0 k_0^{-1} A \exp[-\omega_* g t + \omega_* \xi c_0^{-1} (x - f \xi^{-1} c_0 t) + \pi + \gamma], \tag{1.7}$$

where $\chi = \pm f - ig$ is one of the two pairs of roots of (1.5) and the coefficients A and γ are determined from the relations

$$A = \xi (s_1^2 + f^2 s_2^2)^{\frac{1}{2}} [f^2 s_2^2 + (s_1 - \xi^2)^2]^{-\frac{1}{2}},$$
$$\operatorname{tg} \gamma = [(f^2 + g^2) s_2 - \xi^2 s_1 + n_*^2 (f^2 + g^2)^2] (f \xi^2 s_2)^{-1}, \tag{1.8}$$
$$s_1 = g + n_* (f^2 - g^2), \quad s_2 = 1 - 2 n_* g.$$

We have for system (1.3) from (1.6)

$$u = u^0 \exp[-\omega_* l t + \omega_* \xi c_0^{-1} (x - h \xi^{-1} c_0 t)],$$
$$\theta = u^0 \varkappa T_0 c_0 k_0^{-1} B \exp[-\omega_* l t + \omega_* \xi c_0^{-1} (x - h \xi^{-1} c_0 t) + \pi + \delta], \tag{1.9}$$

where $\chi = \pm h - il$ is one of the two pairs of roots of (1.6) and the coefficients B and δ are determined from the relations

$$B = \xi (h^2 + l^2)^{\frac{1}{2}} [h^2 p_2^2 + (p_1 - \xi^2)^2]^{-\frac{1}{2}},$$
$$\operatorname{tg} \delta = [l (p_1 - \xi^2) + h^2 p_2] [h (l p_2 + \xi^2 - p_1)]^{-1}, \tag{1.10}$$
$$p_1 = l + n_* (h^2 - l^2), \quad p_2 = 1 - 2 n_* l.$$

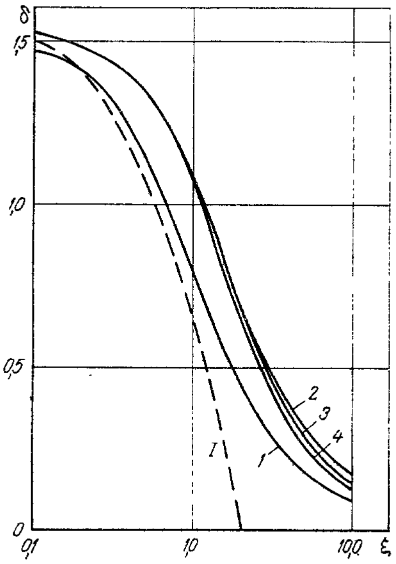

Fig. 4. Dependence of the shift angle δ (rad) on the dimensionless wavelength $\xi = c_0 \eta \omega_*^{-1}$; $e = 1.14 \cdot 10^{-2}$: 1) $n_* = 0.01$, $m_* = 0.5$ (5.0; 10.0); 2) $n_* = 0.5$, $m_* = 0.5$, 3) $n_* = 0.5$, $m_* = 5.0$; 4) $n_* = 0.5$, $m_* = 10.0$. The dashed line I corresponds to a change in the shift angle γ from [8].

If $m_* = n_*$, then $h = f$, $l = g$, $p_i = s_i$ $(i = 1, 2)$, but the amplitudes of the displacement and the temperature θ are not equal to the accuracy of the models (1.1) and (1.3), and the phase shifts γ and δ are also not equal. For example, in particular, for $m_* = n_* \neq 0$

$$(A/B)^2 = 1 + n_* [n_* (f^2 + g^2) - 2g^2 (g + 2n_* f^2)(f^2 + g^2)^{-1}]. \tag{1.11}$$

2. Asymptotic Formulas for Waves with a Fixed Frequency

Let the roots of Eq. (1.6) for $e = 0$ be denoted as follows:

$$\xi_{1,3}^0 = \pm \chi, \quad \xi_{2,4}^0 = \pm (n_* \chi^2 + i\chi)^{\frac{1}{2}}, \tag{2.1}$$

where $\mathrm{Im}\,\xi_2^0 > 0$ has been chosen. Then for $\chi \ll 1$ we obtain from (2.1)

$$c_1 = c_0, \quad q_1 = 0, \tag{2.2}$$

$$c_2 = c_0 (2\chi)^{\frac{1}{2}} + O(\chi^{\frac{3}{2}}), \quad q_2 = q_0 \left(\frac{1}{2} \chi\right)^{\frac{1}{2}} + O(\chi^{\frac{3}{2}}). \tag{2.3}$$

An elastic wave hence corresponds to the root ξ_1^0 and a thermal wave to ξ_2^0.

For $e > 0$ we denote by ξ_1 that root of (1.6) which has the asymptotic form (2.2) for $\chi \ll 1$ and $e \to 0$, and by ξ_2 that root which has the asymptotic form (2.3) under the same conditions. Then the wave corresponding to the root ξ_1 will behave as an elastic wave for $\chi \ll 1$, while the wave corresponding to the root ξ_2 will behave as a thermal wave. Let us consider ξ_i $(i = 1, 2)$ continuous and differentiable functions of χ. Under these hypotheses for $e \neq 0$ and $\varkappa \ll 1$ we obtain the following asymptotic formulas:

$$c_1 = c_0 (1 + e)^{\frac{1}{2}} + O(\chi^2), \tag{2.4}$$

$$q_1 = \frac{1}{2} q_0 e (1 + e)^{-\frac{5}{2}} [1 + (m_* - n_*)(1 + e)] \chi^2 + O(\chi^4),$$

$$c_2 = c_0 [2\chi (1 + e)^{-1}]^{\frac{1}{2}} + O(\chi^{\frac{3}{2}}), \tag{2.5}$$

$$q_2 = q_0\left[\frac{1}{2}\chi(1+e)\right]^{\frac{1}{2}} + O(\chi^{\frac{3}{2}}).$$

For $\chi \gg 1$ and $n_* \neq 0$

$$c_{E,T} = c_0 a_{1,2}^{-\frac{1}{2}} + O(\chi^{-1}),$$
$$q_{E,T} = \frac{1}{2} q_0 b_{1,2} a_{1,2}^{-\frac{1}{2}} + O(\chi^{-1}), \tag{2.6}$$
$$a_{1,2} = \frac{1}{2}\{1 + n_* + em_* \mp [(1 + n_* + em_*)^2 - 4n_*]^{\frac{1}{2}}\},$$
$$b_{1,2} = \mp [(1+e)a_{1,2} - 1][(1 + n_* + em_*)^2 - 4n_*]^{-\frac{1}{2}}. \tag{2.7}$$

Here c_E and q_E denote the asymptotic phase velocity and damping coefficient of the modified elastic wave which correspond as $e \to 0$ to the root ξ_1^0, while c_T and q_T are the same quantities corresponding to the root ξ_2^0 under the same assumptions. If

$$c_1 = c_E,\ q_1 = q_E,\ c_2 = c_T,\ q_2 = q_T \tag{2.8}$$

is satisfied, then the wave with phase velocity c_1 is elastic for both $\chi \ll 1$ and $\chi \gg 1$. The corresponding wave with phase velocity c_2 is thermal in nature in the whole domain.

If

$$c_1 = c_T,\ q_1 = q_T,\ c_2 = c_E,\ q_2 = q_E, \tag{2.9}$$

then a wave with phase velocity c_1 is elastic for $\chi \ll 1$ and thermal for $\chi \gg 1$. On the other hand, the wave with phase velocity c_2 is thermal for $\chi \ll 1$ and elastic for $\chi \gg 1$. Which of the cases (2.8) or (2.9) is actually realized is determined by the properties of the medium. This will be discussed below (Sec. 3). Let us note that the particular case $n_* = 0$, $\chi \gg 1$ will result in the formulas

$$c_E = c_0(1 + em_*)^{-\frac{1}{2}} + O(\chi^{-1}), \tag{2.10}$$
$$q_E = \frac{1}{2} q_0[(1+e)(1+em_*) - 1](1 + em_*)^{-\frac{3}{2}} + O(\chi^{-1}),$$
$$c_T = c_0[2(1 + em_*)\chi]^{\frac{1}{2}} + O(\chi^{-\frac{1}{2}}), \tag{2.11}$$
$$q_T = q_0\chi^{\frac{1}{2}}[2(1 + em_*)]^{-\frac{1}{2}} + O(\chi^{-\frac{1}{2}}).$$

3. Analysis of the Results

Results of calculating the roots of (1.6) with an appropriate passage over to physical quantities are shown in Figs. 1-4. The calculations were performed by using the connectedness coefficient e of steel ($e = 1.14 \cdot 10^{-2}$) and of polyvinyl butyral ($e \approx 0.432$) [11]. The actual numerical values of the parameters n_* and m_* are not known because the physical quantities governing these parameters have either not been determined with sufficient accuracy (the relaxation time of the heat flux τ_0), or have generally not been determined (the relaxation time α). Hence, the calculations have been performed in this paper with several numerical values of τ_0 and α in order to estimate the possible effects caused by these physical parameters.

Let us examine the behavior of fixed-frequency waves. Graphs of the phase velocities c_i are represented in Figs. 1a and b for $e = 1.14 \cdot 10^{-2}$ and $e = 0.432$, respectively. The basic distinctions between the curves obtained, as compared with the curves corresponding to the case of the parabolic heat-conduction equation [7], occur in the frequency range commensurate with the characteristic frequency ω_* or higher. It is understood that we obtain the case of the parabolic heat-conduction equation [7] considered by Chadwick, for $m_* = n_* = 0$. The situation changes for $m_* \geq n_* > 0$. Depending on the value of the connectedness coefficient e and the parameters m_* and n_*, either condition (2.8) or (2.9) can be satisfied. In general, condition (2.8) is always conserved for small e and n_*; i.e., the nature of the wave does not change. For large n_* condition (2.9) is satisfied. It is seen in Fig. 1a ($e = 1.14 \cdot 10^{-2}$) that condition (2.8) is satisfied for $n_* = 0$ and $n_* = 0.5$, and condition (2.9) for $n_* = 0.9$. As in Fig. 1b ($e = 0.432$), for large connectedness coefficients e condition (2.8) is satisfied only for small values of m_* (0.0 and 0.5). Condition (2.9) is satisfied for values of $m_* = 1.0$. As a rule, an abrupt change occurs in the range $0.1 \leq \chi \lessdot 1.0$. The physical meaning of this phenomenon is that upon compliance with (2.9), that wave which is elastic for $\chi \ll 1$ is always the fast wave, while that which is thermal in nature for $\chi \ll 1$ is slow, but both cases can be observed upon compliance with (2.8). In principle, this agrees with the Aschenbach results [12].

The damping coefficients q_i corresponding to the phase velocities in Fig. 1a are represented in Fig. 2 for different values of m_* and n_*. Characteristic here is the fact that the fundamental qualitative measurements occur, compared to the parabolic equation case [7], in the frequency range $\chi > 1.0$, i.e., above the characteristic frequency.

The connectedness coefficient B and the phase-shift angle δ are characteristic for fixed-frequency waves. The roots of (1.6) to determine B and of (1.10) to determine δ correspond to a modified elastic wave as in [7] (Figs. 3 and 4, respectively). As in [8], the main distinction from classical theory [7] is observed for $\xi \sim 1.0$. The change turns out to be not very substantial for the phase-shift angle (Fig. 4). The dashed lines I in Figs. 3 and 4 show the curves for changes in these quantities determined by means of model (1.1) [8]. As has been shown above, they differ substantially from the curves obtained by means of model (1.3).

Therefore, taking account of the rate of temperature change in the equations of state to the accuracy of model (1.3) is, just as taking account of heat-flux relaxation, necessary at high frequencies (on the order of the characteristic frequencies) or for small wavelengths. The dependences obtained can be used to check experimentally determined relaxation parameters.

NOTATION

$u(x, t)$, displacement; T, T_0, linear and initial temperature; λ, μ, Lamé constants; ρ, ρ_0, linear and initial densities of the medium; α_T, coefficient of thermal expansion; τ_0, relaxation time of the heat flux; k_0, heat-conduction coefficient; c_e, specific heat per unit mass; S, specific entropy; r, specific heat source intensity; q_i, heat flux components; u^0, θ^0, plane wave amplitudes; η, wavelength; ω, wave frequency.

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